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# KEPLERIAN MOVEMENT AND HARMONIC OSCILLATORS

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# KEPLERIAN MOVEMENT AND HARMONIC OSCILLATORS

By Claude A. Burdet in Zurich

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## Introduction

For some years, a group of methods have been found under the term <u>regularization</u>, whose aim is to eliminate the singularity from the field of gravitational force in $r = 0$. By various transformations, such efforts permit replacing the equations called "classic" from the movement

$$\ddot{\vec{x}} + \frac{M}{r^3}\,\vec{x} = \vec{P}, \tag{1}$$

where $r$ is the distance $r^2 = \sum_{i=1}^{3} x_i^2$ and $\left(\dot{\ } = \frac{d}{dt}\right)$, describing the problem of the two bodies subjected to any disturbing force $\vec{P}\left(\vec{x}, \dot{\vec{x}}, t\right)$.

By virtue of the extreme stability (analytical and numerical) and still other properties which we shall expose in what follows, differential <u>linear equations with constant coefficients</u> (harmonic oscillators) are particularly attractive and one will naturally try to return (1) to this type. A similar result has been established with success in [1] then in [2], [3].

The first paragraph is devoted to a brief presentation of the method described in greater detail in [3]. Then, in paragraph 2 a new transformation of time $t$ furnishes another harmonic oscillator whose principle advantage is also to permit eliminating the $1/r$ singularity which appears with certain disturbing forces (especially those due to the flatness of the Earth).

## 1. The Central Method

The solutions of the non-disturbed system

$$\ddot{\vec{x}} + \frac{M}{r^3}\,x = 0 \tag{2}$$

are conics whose center is at the origin.

The central method consists of determining such orbits (conics) with the help of a harmonic oscillator fixed at the center of the conic. The square of the frequency $w^2$ determines the type of conic

$$w^2 > 0\text{: ellipses,}$$

$$w^2 = 0\text{: parabolas,}$$

$$w^2 < 0\text{: hyperbolas.}$$

Tridimensional harmonic oscillator: Let's designate by C the coordinates of the center of the oscillator and the position of the material point by x in a system of cartesian coordinates. As a function of an independent variable s, the oscillating movement is then governed by the differential equations

$$(3) \qquad \vec{x}'' + w^2\left(\vec{x} - \vec{C}\right) = 0,$$

where $' = \frac{d}{ds}$, $w^2 \lesseqgtr 0$.

It is easy to establish that the material point describes a conic and that the origin $\vec{x} = 0$ is located in its center.

Now let's transform the disturbed system (1) into a system of type (3) by introducing an imaginary central time $s_1$ as an independent variable, defined beginning with physical time t by:

$$(4) \qquad dt = r\,ds_1.$$

With the help of the Laplace Vector

$$(5a) \qquad \vec{A} = \left(v^2 - \frac{1}{r}\right)\vec{x} - \left(\vec{x}, \dot{\vec{x}}\right)\dot{\vec{x}} \qquad ^{1)},$$

and Keplerian energy

$$(5b) \qquad w^2 = \frac{2}{r} - v^2, \text{ where } v^2 = \sum_{i=1}^{3} \dot{x}_i^2,$$

1) The notation ($\vec{x}$, $\vec{y}$) designates the scaler of the vectors $\vec{x}$ and $\vec{y}$.

(We have posed the constant $\underline{M}$ equal to 1 without loss of generality) the system (1) allows itself to be brought back to

$$\boxed{\begin{aligned} &(6a) \quad \vec{x}'' + w^2\vec{x} + \vec{A} = r^2\vec{P}, \\ &(6b) \quad t' = r. \end{aligned}}$$

The detail of transformation presents no difficulties and can be found again in [3].

If it is remembered that the quantities $\underline{w^2}$ and $\underline{\vec{A}}$ are integrals (= elements for constants) of nondisturbed Keplerian movement, the goal has been reached since these constants calculated beginning from initial conditions. The oscillator

$$\vec{x}'' + w^2\vec{x} + \vec{A} = 0$$

describes a conic of

$$(7) \qquad \begin{aligned} &\text{half of the full axis} \quad a = \frac{1}{w^2}, \\ &\text{center} \quad \vec{c} = -\frac{1}{w^2}\vec{A}, \\ &\text{eccentricity} \quad e = |\vec{A}|. \end{aligned}$$

But the oscillator (6) is disturbed $\left(\underline{\vec{P}} \neq 0\right)$; therefore we must still determine $\underline{w^2}$ and $\underline{\vec{A}}$ by simultaneously integrated (8) with (6):

$$\boxed{\begin{aligned} &(8a) \quad \vec{A}' = 2\left(\vec{P}, \vec{x}'\right)\vec{x} - \left(\vec{x}, \vec{x}'\right)\vec{P} - \left(\vec{P}, \vec{x}\right)\vec{x}', \\ &(8b) \quad (w^2)' = -2\left(\vec{P}, \vec{x}'\right). \end{aligned}}$$

The system (6), (8) constitutes a system of order 11 for integration of the disturbed Keplerian movement. This raised order should not cause concern, for the experiment shows that the order of the system plays a much less important role than the type (here linear) of the differential equations, concerning the exactness of numerical integration.

The introduction of the imaginery time $\underline{s}_1$ plays, in addition to, the role of an automatic control, as a function of the distance $\underline{r}$, of the value of the integration step $\Delta \underline{t} = \underline{r}\,\Delta\,\underline{s} = \underline{rh}$ where $\underline{h}$ is the numerical step.

On the other hand, let's mention here that while this does not constitute an advantage for practical cases, that this method also permits integration of a collision ($\underline{r} = 0$).

In Appendix I, the reader will find the elements which can be introduced for the central method, and the differential equations which they satisfy, thus permitting replacing the integration of the coordinates $\vec{x}$ and of the time $\underline{t}$. In a general way, the integration of elements is to be advised, especially when the disturbing forces are weak. Finally let's cite the differential equation for the distance $\underline{r}$

(9) $$\boxed{r'' + w^2 r - 1 = r\left(\vec{P}, \vec{x}\right)}$$

which is also of type (3) and which renders valuable services for the integration of the time with the help of an element (Appendix I).

## 2. The Focal Method

The relative position of the two bodies which we have represented by the vector $\vec{x}$ can also be characterized by the distance $\underline{r}$ and the unit vector $\vec{y}$ indicating the direction of $\vec{x}$. This permits defining a new harmonic oscillator characterizing the variables $\vec{y}$ and $\underline{u} = \frac{1}{r}$; its center is found in $\vec{x} = 0$, in other words at the center of gravitational attraction which is also the center of the solution conics of the system (1).

In order to do this, let's define again beginning from time $\underline{t}$ and imaginery focal time $s_2$ by:

(10) $$dt = r^2 ds_2 .$$

Since $\vec{y} = \frac{1}{r}\vec{x}$, one has $\ddot{\vec{x}} = (\vec{y} \cdot r)^{\cdot\cdot} = r\ddot{\vec{y}} + 2\dot{r}\dot{\vec{y}} + \ddot{r}\vec{y}$ and the system (1) becomes (while still placing $\underline{M} = 1$)

(11) $$\ddot{\vec{y}}r + 2\dot{r}\dot{\vec{y}} + \ddot{r}\vec{y} + \frac{1}{r^2}\vec{y} = \vec{P}.$$

On the other hand

$$= \frac{d}{dt} = \frac{1}{r^2}\frac{d}{ds_2},$$

$$\ddot{} = \frac{d^2}{dt^2} = \frac{1}{r^4}\frac{d^2}{ds_2^2} - \frac{2r'}{r^5}\frac{d}{ds_2}, \left(\text{where } ' = \frac{d}{ds_2}\right)$$

involves

(12) $$\vec{y}'' - \frac{2r'}{r}\vec{y}' + \left[\frac{r''}{r} - 2\left(\frac{r'}{r}\right)^2 + r\right]\vec{y} + \frac{2r'}{r}\vec{y}' = r^3\vec{P},$$

$\underline{\vec{y}'}$ compensate each other and disappear.

Now let's multiply (1) scalarly by $\underline{\vec{x}}''$

(13) $$\left(\ddot{\vec{x}}, \vec{x}\right) + \frac{1}{r} = \left(\vec{P}, \vec{x}\right)$$

in order to obtain

(14) $$r\ddot{r} - v^2 + \dot{r}^2 + \frac{1}{r} = \left(\vec{P}, \vec{x}\right)$$

since $\underline{r}^2 = (\underline{\vec{x}}, \underline{\vec{x}})$ implies

$$r\ddot{r} + \dot{r}^2 = \left(\vec{x}, \ddot{\vec{x}}\right) + \left(\dot{\vec{x}}, \dot{\vec{x}}\right).$$

Finally, let's introduce $\underline{s}_2$ as an independent variable in (14)

(15) $$\frac{r''}{r} - 2\left(\frac{r'}{r}\right)^2 + r - p = r^2\left(\vec{P}, \vec{x}\right),$$

with

(16) $$p = r^2(v^2 - \dot{r}^2) = (\vec{y}', \vec{y}').$$

from (15) and (12) onedraws

(17a) $$\boxed{\vec{y}'' + p\vec{y} = r^3\left(\vec{P} - \left(\vec{P}, \vec{y}\right)\vec{y}\right),}$$

(17b) $$\boxed{u'' + pu - 1 = -r^2\left(\vec{P}, \vec{y}\right),}$$

(17c) $$\boxed{t' = r^2,}$$

if one places $\underline{u} = \frac{1}{\underline{r}}$ in (15).

Again, while noting that $\underline{p}$ is an $\underline{\text{element}}$ of nondisturbed Keplerian movement, one can recognize in (17) a harmonic oscillator of frequency $\underline{\sqrt{p}}$. And

And since we have a disturbing force $\vec{P} \neq 0$, the differential equation must still be added

(18) $$\boxed{p' = 2r^3 (\vec{P}, \vec{y}'),}$$

obtained by differentiating (16).

The system (17), (18) is of order 10, and its simultaneous integration furnishes the disturbed Keplerian movement desired.

The same remarks can be made here as for the central method concerning the precision of numerical integration, since the differential equations are similar in all points. Automatic control of the step varies herewith $r^2$, and Appendix II indicates how the adequate elements can be introduced. However, some pecularities deserve to be emphasized for the focal method:

The parameter $p$ is $\geqq 0$ for all types of conics, $p = 0$ corresponding to rectilinear movement.

The focal method does not permit integration of a collision, for an infinite time $s_2$ would be necessary in order to arrive at the collision itself ($r = 0$).

When the disturbing force $P$ is in the form

$$\vec{P} = \sum_{v} \frac{1}{r^v} \vec{Q}_v$$

the right members (disturbances) of the oscillators of (17) take the form

$$\sum_{v} u^v \cdot \vec{Q}_v$$

which means that the denominators $\frac{1}{r^v}$ have become polynomials $u^v$ of trigonometric functions in $s_2$. That is an advantage of this method, for the case where $\vec{P}$ includes the disturbance due to the flatness of the Earth, for example.

The order of(17) (18) is 10 but it is necessary to take into account the identities

$$(\vec{y}, \vec{y}) = 1, \ (\vec{y}, \vec{y}') = 0, \text{ and } p = (\vec{y}', \vec{y}'),$$

which permit reducing it and can render remarkable service in the calculation of the right members (cf. example).

Example: Disturbance due to the flatness of the Earth. We illustrate the focal method by this example where we shall limit development in Legendre polynomials of the disturbing forces, to the initial terms (those of $J_2$); the terms of higher order will be treated in an identical manner. Therefore let the force be

$$\vec{P} = \frac{c}{r^4} \left[ f(z) \vec{y} + g(z) \vec{i}_3 \right],$$

where

$$f(z) = \frac{1}{2} (15 y_3^2 - 3),$$

$$g(z) = -3y_3,$$

$$c = \text{constant}.$$

$\vec{i}_3$ = unit vector of axis $x_3$ (toward the north), the plane $(x_1, x_2)$ being equatorial.

For the right members of (17) (18), one has

$$\text{(17a)} \qquad r^3 \left( \vec{P} - \left( \vec{P}, \vec{y} \right) \vec{y} \right) = \frac{c}{r} (g \vec{i}_3 - g y_3 \vec{y}) = cgu (\vec{i}_3 - y_3 \vec{y})$$

$$\text{(17b)} \qquad -r^2 \left( \vec{P}, \vec{y} \right) = -\frac{c}{r^2} (f + g y_3) = -cu^2 (f + y_3 g) = \frac{3}{2} cu^2 (1 - 3y_3^2)$$

$$\text{(18)} \qquad r^3 \left( \vec{P}, \vec{y}' \right) = \frac{c}{r} (f(\vec{y}, \vec{y}') + g y'_3) = cugy'_3.$$

## 3. Conclusions

In this method, the reader will be able to find again the most significant characteristics of the two systems of differential equations (central and focal) which permit replacing the "classic" equations, for the integration of the general problem of the two bodies.

All the differential equations governing nondisturbed Keplerian movement are linear, with constant coefficients (these coefficients are elements of the movement) and the disturbed movement can therefore be treated with the help of the variation method of the constants.

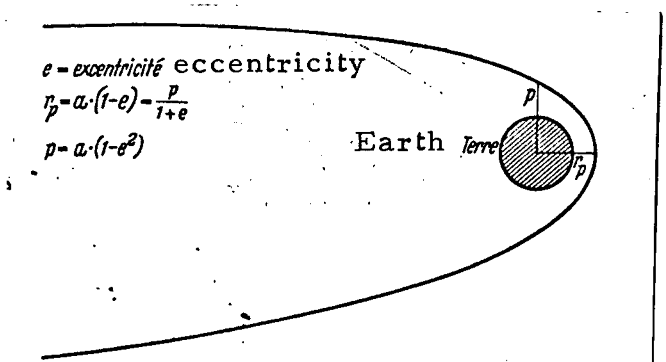


Figure 1. Passage autour de la Terre (orbite très excentrique)

Figure 1. Passage around the Earth (very eccentric orbit)

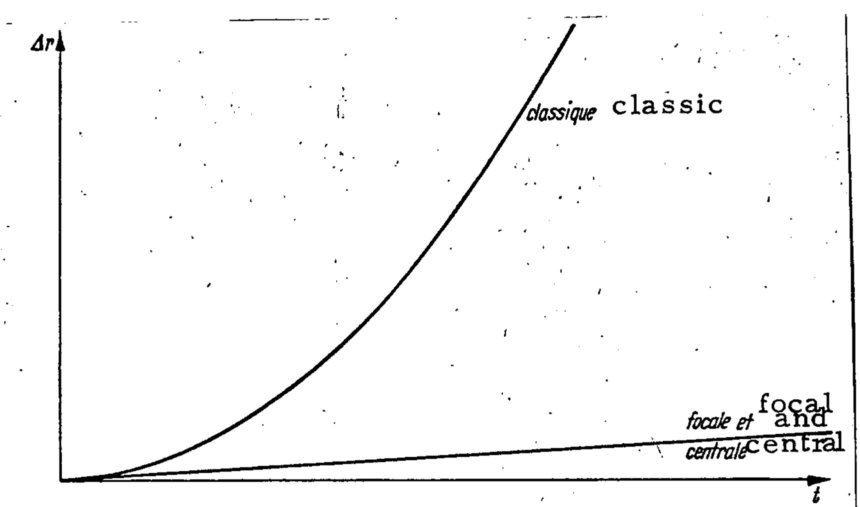


Figure 2. Comparaison des erreurs d'intégration numérique (orbite circulaire)

Figure 2. Comparison of the errors of numerical integration (circular orbit)

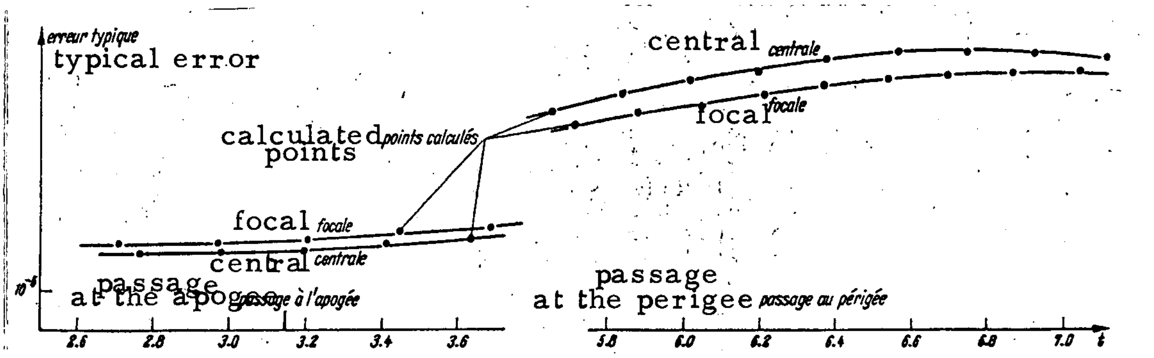


Figure 3. Comparaison des erreurs d'intégration numérique (orbite légèrement excentrique)

Figure 3. Comparison of the errors of numerical integration (slightly eccentric orbit)

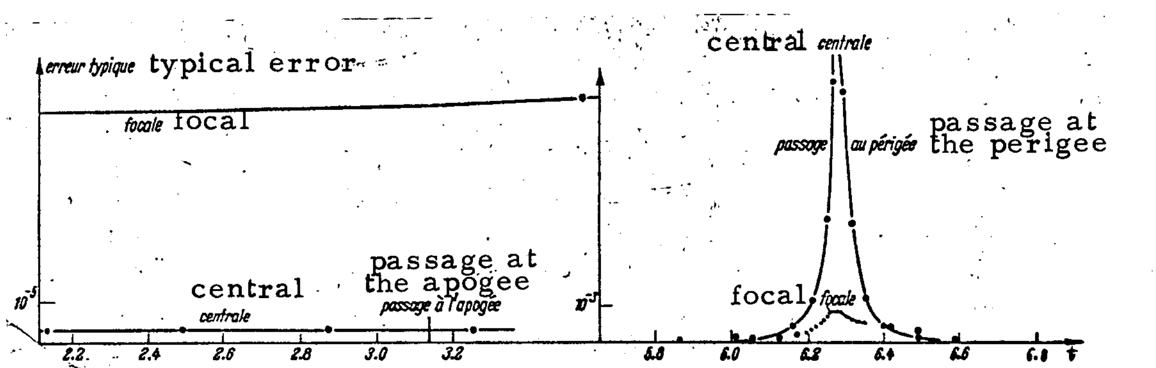


Figure 4. Comparaison des erreurs d'intégration numérique (orbite fortement excentrique)

Figure 4. Comparison of the errors of numerical integration (highly eccentric orbit)

Thus, one can define in a mathematically natural manner, new elements for Keplerian movement which enjoy geometric and mechanical properties common to all types of orbit (circular, elliptical, rectilinear, parabolic and hyperbolic) so well that all these cases can be treated by a single set of formulas. At the time of a disturbed Keplerian movement, these elements satisfy, in addition, differential equations of simple form which thus permit the use of an economical and precise integration method of the disturbances of the elements. The integration of the elements is still to be advised in regard to the integration of the coordinates ($\vec{\underline{x}}$, $\underline{t}$), unless the disturbing forces are not very large.

The use of a fixed reference system with cartesian coordinates guarantees the application of these methods (central and focal) in all cases, without fear of critical values for the inclination of the orbit, or numerical indetermination of the formulas utilized. On the other hand, the disturbing forces $\vec{\underline{P}}$ do not undergo any transformation and can be introduced just as they are. Orbits of all types can be integrated (also numerically) without fear of degeneration. The integration of a collision with the central body is, however, only possible with the central method.

For a numerical integration, the transformations of the independent variable possess a supplementary advantage since they introduce, in the differential equations themselves, an automatic control of the numerical step of integration, as a function of the distance $\underline{r}$ (or $\underline{r}^2$). The schemas opening out on several steps (multi-step methods) therefore become particularly advantageous since it is no longer necessary to modify the numerical value of the integration step.

From the analytical viewpoint, the focal method possesses the pecularity of representing the quantity $\frac{1}{\underline{r}} = \underline{u}$ by a simple trigonometric expression. For the case where the disturbing forces are due to the flatness of the Earth, for example, that means that one can develop these forces (which contain $\frac{1}{\underline{r}^n}$ factors) in a series of Fourier <u>finites</u> (Fourier polynomials). By virtue of the multiple possibilities offered by differential linear equations with constant coefficients, such disturbances can therefore be integrated directly without too much difficulty.

<u>The frequencies of the central method $\omega$ and focal method $\sqrt{p}$.</u> Without making a detailed study of this type of movement, it is obvious that the frequency

of a harmonic oscillator is a principle source of instability: in error, no matter how small, affecting the value of the frequency leads, after a certain time lapse, to diametrically opposed results in integration. Besides, this fact is to be placed in relation with the orbital stability of the Keplerian movement [3]. It is suitable, therefore, to proceed to the determination of the quantities $\underline{w}^2$ and $\underline{p}$ with most particular care.

If the user decides to take this state of fact into account, he can make use of the following processes:

a) Utilization of a more precise method of numerical squaring for the frequency than for the other quantities. This increases precision without perceptibly changing calculation times.

b) Integration of the frequency by a procedure of the type recommended by Encke for "classic" equations. One is given (in the first approximation) a function $\underline{\tilde{w}}^2\ (\underline{s}_1)$; next, beginning with $\underline{\tilde{w}}^2\ (\underline{s}_1)$ and with the Keplerian osculatory frequency $\underline{w}^2$ one defines:

$$\Delta w^2 := \tilde{w}^2\ (s_1) - w^2\ (s_1).$$

This quantity $\Delta \underline{w}^2$ can therefore be determined by integration of

$$\frac{d}{ds_1}\ (\Delta w^2) = 2\left(\vec{P}, \vec{x}'\right) + \frac{d\tilde{w}^2}{ds_1}$$

and, in the measure where $\underline{\tilde{w}}^2\ (\underline{s}_1)$ has been chosen correctly, the quantity $\Delta \underline{w}^2$ is of the second order.

The oscillator itself becomes

$$\begin{aligned} \vec{x}'' + \tilde{w}^2\vec{x} = \vec{A} &= r^2\vec{P} = \Delta w^2\vec{x}, \\ r'' + \tilde{w}^2 r - 1 &= r\left(\left(\vec{P}, \vec{x}\right) + \Delta w^2\right), \end{aligned}$$

where one will notice that the base frequency $\underline{\tilde{w}}$ is better adapted to the disturbed movement $\underline{w}$.

In like manner, one will have for the focal method, with $\tilde{p}(\underline{s_2})$:

$$\Delta p := \tilde{p}(s_2) - p(s_2),$$

where $\underline{p}$ is the Keplerian parameter of the osculatory orbit. Moreover

$$\frac{d}{ds_2}\Delta p = \frac{d\tilde{p}}{ds_2} - 2r^3\left(\vec{P}, \vec{y}'\right)$$

with

$$\vec{y}'' + \tilde{p}\vec{y} = r^3\vec{P} + \left(\Delta p - r^3\left(\vec{P}, \vec{y}\right)\right)\vec{y},$$

$$u'' + \tilde{p}u - 1 = \Delta pu - r^2\left(\vec{P}, \vec{y}\right).$$

A particular case of this process still deserves being analyzed in more detail. It is the case

$$\tilde{w}^2 = \text{constant (or } \tilde{p} = \text{constant).}$$

The experiment shows indeed that (numerically as well as analytically), this special case leads to better results. Moreover, the important practical case where the disturbing forces (or a part of these) are <u>conservative</u> is exactly this type.

Therefore let the forces be $\vec{P} = -\frac{\partial V}{\partial \vec{x}}$, where V is a conservative potential. One has

$$\frac{d}{ds_1}\left(\Delta w^2\right) = 2\left(\vec{P}, \vec{x}'\right) = -2\frac{dV}{ds_1},$$

or

$$\Delta w^2 = -2V.$$

Therefore

$$\tilde{w}^2 = w^2_{\text{Képler}} + \Delta w^2 = -2E_{\text{Képler}} - 2V = -2E_{\text{tot}} = \text{constant},$$

since the disturbance is conservative; for the oscillator one has, in addition,

$$\vec{x}'' + \tilde{w}^2\vec{x} + \vec{A} = -\frac{\partial}{\partial x}(r^2V),$$

$$r'' + \tilde{w}r - 1 = -\frac{\partial}{\partial r}(r^2V).$$

This is equally possible for the focal method, and one has:

$$\vec{P} = -\frac{\partial V}{\partial x} = -u\left(\frac{\partial V}{\partial \vec{y}} - \left(\frac{\partial V}{\partial \vec{y}}, \vec{y}\right)\vec{y}\right) + u^2\frac{\partial V}{\partial u}\vec{y}.$$

On the other hand, this particular case permits differentiating the forces $\underline{\vec{P}}$ and one will then place

$$\tilde{p} := p + \Delta p + r^3\left(\vec{P}, \vec{y}\right) = p + \Delta p + r\frac{\partial V}{\partial u} = \text{constant},$$

Therefore

$$\frac{d}{ds_2}(\Delta p) = 2r^3\left(\frac{\partial V}{\partial y}, \vec{y}'\right) - \frac{d}{ds_2}\left(r\frac{\partial V}{\partial u}\right),$$

with

$$\vec{y}'' + \tilde{p}\vec{y} = -r^2\frac{\partial V}{\partial \vec{y}} + \Delta p\vec{y},$$

$$u'' + \tilde{p}u - 1 = \Delta pu.$$

Here one will be able to choose as a value of $\underline{\tilde{p}}$ = constant the average value is the time $\underline{\bar{\tilde{p}}}$, for example.

To conclude, let's mention that the central method permits regularizing the restricted problem of the three bodies. Moreover, if one admits an elliptical movement for the Moon, one must then apply the focal method to this movement. And one can, therefore, without any other difficulty, consider a disturbed movement for the Moon (flatness of the Earth, solar attraction, etc...) as well as supplementary disturbances (flatness of the Earth, attraction of the sun, resistance of the air, solar radiation, etc...) for the satellite of negligable mass. The theoretical and experimental results of this problem depart somewhat from the limits of this article and are not presented here.

## Appendices

The interested reader, by a direct application of the methods exposed above, will find, in what follows, the necessary basis for a concrete application.

The intermediary calculations have been omitted, but they present no major difficulties and can be easily found again. For each of the two methods, one will find, in the first place, the systems of differential equations for the osculatory elements. Next an equivalent system, but of simpler form, is described.

Appendix I: The elements of the central method. In order to obtain osculatory elements, it is necessary to depart from the Keplerian frequency $w$ and from the central oscillator which is associated with it:

$$\vec{x}'' + w^2\vec{x} + \vec{A} = \vec{f} = r^2\vec{P},$$

$$r'' + w^2 r - 1 = f_4 = r\left(\vec{P}, \vec{x}\right),$$

$$t' = r.$$

In order to take all the cases into account $w^2 > 0$, $w^2 < 0$, $w^2 = 0$ let's place

$$\vec{x} = \vec{\alpha}(s_1)B2(w^2, s_1) + \vec{\beta}(s_1)\, B1(w^2, s_1) + \vec{\gamma}(s_1),$$

$$\vec{x}' = \vec{\alpha}(s_1)B1(w^2, s_1) + \vec{\beta}(s_1)\, B0(w^2, s_1),$$

$$r = \delta_1(s_1)B2(w^2, s_1) + \delta_2(s_1)B1(w^2, s_1), + \delta_3(s_1),$$

$$r' = \delta_1(s_1)B1(w^2, s_1) + \delta_2(s_1)B0(w^2, s_1),$$

$$t = \delta_1(s_1)B3(w^2, s_1) + \delta_2(s_1)B2(w^2, s_1) + \delta_3(s_1)s_1 + \delta_0(s_1),$$

with $\delta_3 = \dfrac{1 - \delta_1}{w^2}$ and $\vec{y} = \dfrac{-1}{w^2}\left(\vec{A} + \vec{\alpha}\right)$.

The special functions $B(q, s)$ are regular and well defined for all the real values of $q$ and $s$; they are defined by:

$$B0(q, s) = \cos\sqrt{q}s,$$

$$B1(q, s) = \frac{1}{\sqrt{q}}\sin\sqrt{q}s,$$

$$B2(q, s) = \frac{1}{q}(1 - B0(q, s)),$$

$$B3(q, s) = \frac{1}{q}(s - B1(q, s)),$$

$$B4(q, s) = \frac{1}{q}(B3(q, s) + B1(q, s)\, B2(q, s)),$$

$$B5(q, s) = \frac{1}{q}(3\, B3(q, s) - B1(q, s)\, B2(q, s)).$$

By a calculation which can be found again in [3] one finally obtains

$$
\begin{aligned}
(w^2)' &= -2\,C1, \\
\vec{\alpha}' &= (r^2 w^2\, B1 + C2\, B0)\,\vec{P} + (C3\, B0 - C1\, B1)\,\vec{x}' - C1\, s\,\vec{\beta}, \\
\vec{\beta}' &= (r^2\, B0 - C2\, B1)\,\vec{P} - C3\, B1\,\vec{x}' + B1^2\, C1\,\vec{\beta} + C1\,\widetilde{B3}\,\vec{\alpha}, \\
\vec{\gamma}' &= (C2\, B2 - r^2\, B1)\,\vec{P} + C3\, B2\,\vec{x}' + C1\,(B2^2\,\vec{\alpha} + (B3 - B1\, B2)\,\vec{\beta}), \\
\delta_1' &= rw^2\, C3\, B1 - C1\,(w^2(\delta_1\, B2^2 + \delta_2\,\widetilde{B3}) - 2\,\delta_3\, B0), \\
\delta_2' &= rC3\, B0 - C1\,(\delta_1\,(B1\, B2 - B3) + \delta_2\, B1^2 + 2\,\delta_3\, B1), \\
\delta_3' &= -\,rC3\, B1 + C1\,(\delta_1\, B2^2 + \delta_2\,\widetilde{B3} + 2\,\delta_3\, B2), \\
\delta_0' &= rC3\, B2 - C1\,(\delta_1\, B5 + 2\,\delta_3\, B3 + \delta_2\, B2^2),
\end{aligned}
$$

where $C1 = \left(\vec{P}, \vec{x}'\right)$, $C2 = (\vec{x}, \vec{x}') = rr'$, $C3 = \left(\vec{P}, \vec{x}'\right)$, and where all the special functions $\underline{B}$ are taken with proofs $(w^2, s_1)$ with $\widetilde{B3}\,(q, s) = 4\, B3\,(4q, s)$.

These elements still verify the identities

$$
\begin{aligned}
\vec{\gamma} &= \delta_2\vec{\beta} - \delta_3\vec{\alpha}, \\
\delta_1 &= 1 - w^2\,\delta_3 = \left(1 - w^2\left(\vec{\beta}, \vec{\beta}\right)\right)^{\frac{1}{2}}, \\
\delta_2 &= \frac{1}{\delta_1}\left(\vec{\alpha}, \vec{\beta}\right), \\
\delta_3 &= \left(\vec{\beta}, \vec{\beta}\right) / (1 + \delta_1),
\end{aligned}
$$

coming from the relations

$$r^2 = (\vec{x}, \vec{x}) \text{ and } rr' = (\vec{x}, \vec{x}').$$

These identities can, therefore, be utilized to reduce the total order of the system, and consequently the number of squarings, or then even as controls of the precision of numerical integration.

Beginning with this method of integration of osculatory elements, one can deduce a more precise and more economical method (in time of calculation);

for that, it is necessary to consider an oscillator with a constant base frequency $\tilde{w}$

$$\vec{x}'' + \tilde{w}^2\vec{x} + \vec{A} = \vec{f} = r^2\vec{P} + \Delta w^2\vec{x},$$

$$r'' + \tilde{w}^2 r - 1 = f_4 = r\left((\vec{P}, \vec{x}) + \Delta w^2\right).$$

The solution remains similar in all points, only the differential equations change, which become:

$$(\Delta w^2)' = 2\ C1,$$

$$\vec{\alpha}' = \tilde{w}^2 B1\ \vec{f} - B0\ \vec{A}',$$

$$\vec{\beta}' = B0\ \vec{f} - B1\ \vec{A}',$$

$$\vec{\gamma}' = -\ B1\ \vec{f} - B2\ \vec{A}',$$

$$\delta_1' = -\ \tilde{w}^2\,\delta_3' = \tilde{w}^2\ B1\ f_4,$$

$$\delta_2' = B0\ f_4,$$

$$\delta_0' = B2\ f_4,$$

one will replace $\vec{A}'$ in all these equations by the expression

$$\vec{A}' = 2\ C1\ \vec{x} - C2\ \vec{P} - C3\ \vec{x}'.$$

Here also one will make use of the identity

$$\vec{\gamma} = \delta_2\vec{\beta} - \delta_3\vec{\alpha} + r\,\Delta w^2\vec{x}$$

and the relations

$$r^2 = (\vec{x}, \vec{x}) \text{ and } rr' = (\vec{x}, \vec{x}')$$

can also serve as a control.

Remarks. In the form presented, the osculatory method of integration includes from 8 to 14 squarings, according to whether one utilizes the identities among elements or not. The total order of the "classic" system, however, is equal to 6; it is, therefore, obvious that our elements contain a certain redundance.

Indeed, $w^2$ can also be expressed as a function of $\vec{\alpha}$ and of $\vec{\beta}$; and, besides, the eccentricity $e$ can be expressed in two distinct manners as a function of $\vec{\alpha}$ and $\vec{\beta}$, which furnishes a supplementary identity. It is, however, advantageous to renounce these identities, for their use increases the calculation times and introduces a source of needless numerical error.

On the other hand, the experiment has shown that the numerical behavior of the method (errors due to numerical rounding off, calculation times, etc...) is much less sensitive to an artificial increase of the number of squarings than to the use of identities which are mathematically exact but numerically weak.

Moreover, it is very useful to dispose of, in addition to the numerical solution, a certain number of controls which permit estimating its real value.

Appendix II: The elements of the focal method. In a manner highly analogous to the central case, it is necessary, in order to obtain osculatory elements in the focal method, to depart from the Keplerian frequency $\sqrt{p}$ and from the focal oscillator which is associated with it.

$$\vec{y}'' + p\vec{y} = \vec{g} = r^3 \left(\vec{P} - \left(\vec{P}, \vec{y}\right) \vec{y}\right),$$

$$u'' + pu - 1 = g_4 = - r^2 \left(\vec{P}, \vec{y}\right),$$

$$t' = r^2 = \frac{1}{u^2}.$$

In order to take all $p \gtreqless 0$, into account, one defines the solution by:

$$\vec{y} = \vec{\alpha}(s_2)B0(p, s_2) + \vec{\beta}(s_2)B1(p, s_2),$$

$$\vec{y}' = -p\vec{\alpha}(s_2)B1(p, s_2) + \vec{\beta}(s_2)B0(p, s_2),$$

$$u = \delta_1(s_2)B2(p, s_2) + \delta_2(s_2)B1(p, s_2) + \delta_3(s_2),$$

$$u' = \delta_1(s_2)B1(p, s_2) + \delta_2(s_2)B0(p, s_2),$$

with $\delta_3 = \dfrac{1 - \delta_1}{p}$. Then one obtains the equations

$$p' = 2r^3 \left(\vec{P}, \vec{y}'\right),$$

$$\vec{\alpha}' = B1\vec{g} - C1 \left(B1^2 \vec{\alpha} + B4 \vec{\beta}\right),$$

$$\vec{\beta}' = B0\,\vec{g} + C1\left((s_2 + B1\ B0)\vec{\alpha} + p\ B1^2 \vec{\beta}\right),$$

$$\delta_1' = p\ B1\ g_4 + C1\ (\delta_1 p\ B2^2 - 2\delta_3\ B0 + \delta_2 p\ \widetilde{B3}),$$

$$\delta_2' = B0\ g_4 + C1\ (\delta_1(B1\ B2 - B3) + 2\delta_3\ B1 + \delta_2\ B1^2),$$

$$\delta_3' = -\ B1\ g_4 - C1\ (\delta_1\ B2^2 + 2\delta_3\ B2 + \delta_2\ \widetilde{B3})$$

where $C1 = \frac{p'}{2}$ and with $\widehat{B3} = 4\ B3(4p,\ s_2)$, all the special functions $\underline{B}$ being assumed with proofs $(p, s_2)$. And again, one can convert here the identities

$$(\vec{y}, \vec{y}) = 1,\ (\vec{y}, \vec{y}') = 0 \text{ and } (\vec{y}', \vec{y}') = p$$

to osculatory identities

$$w^2 = (1 + \delta_1)\,\delta_3 - \delta_2^2,$$

$$(\vec{\alpha}, \vec{\alpha}) = 1,$$

$$(\vec{\beta}, \vec{\beta}) = p,$$

$$(\vec{\alpha}, \vec{\beta}) = 0.$$

The reduced focal method is produced by a focal oscillator of constant base frequency $\sqrt{\underline{p}}$

$$\vec{y}'' + \tilde{p}\vec{y} = \vec{g} = r^3\,\vec{P} + \left(\Delta p - r^3 \left(\vec{P}, \vec{y}\right)\vec{y}\right),$$

$$u'' + \tilde{p}u - 1 = g_4 = \Delta pu - r^2 \left(\vec{P}, \vec{y}\right).$$

The solution is still expressed by the same expressions as previously for the osculatory focal method, but the differential equations become

$$\Delta p' = -\ 2r^3 \left(\vec{P}, \vec{y}'\right),$$

$$\vec{\alpha}' = = B1\,\vec{g},$$

$$\vec{\beta}' = B0\,\vec{g},$$

$$\delta_1' = -\ \tilde{p}\,\delta_3' = \tilde{p}\ B1\ g_4,$$

$$\delta_2' = B0\ g_4.$$

Up till here we have left aside the integration of physical times $\underline{t}$; the latter can also be brought back to the integration of an element $\delta_0$ in the focal method, as for the central method.

One places for this purpose

$$t = 2\ B6(w^2,\ Z) - \frac{2pX}{Z^2 + w^2} + \delta_0,$$

where

$$w^2 := (1 + \delta_1)\,\delta_3 - \delta_2^2,$$
$$X := B7\left(p,\ \frac{s_2}{2}\right),$$
$$Z := (1 + \delta_1)\,X + \delta_2$$

and one has for $\delta_0$ the differential equation

$$\delta_0' = ww'\left(B8\,(w^2, Z) - \frac{4pX}{(Z^2 + w^2)^2}\right) - 2\left(\frac{p}{1 + \delta_1} + \frac{1}{u}\right)\frac{\delta'_1 X + \delta'_2}{Z^2 + w^2}$$
$$+ \frac{p'}{Z^2 + w^2}\left(2X - \frac{1 + \delta_1}{u}\,B9(p, s_2)\right),$$

the initial conditions being

$$\delta_0(0) = t_0 - 2\ B6\ (w^2(0),\ Z(0)\ ).$$

In the reduced method, where $\underline{\tilde{p}}$ is a constant, this differential equation is brought back to

$$\delta_0' = g_4\left(u'\left(\frac{4\tilde{p}X}{(Z^2 + w^2)^2} - B8\,(w^2, Z)\right) - \frac{2}{Z^2 + w^2}\left(\frac{\tilde{p}}{1 + \delta_1} + \frac{1}{u}\right)\right).$$

The special functions $\underline{B}$ utilized here are also of a type which permits integration of all orbits (even rectilinear, but without collision). By definition, one has

$$B6\ (w^2,\ Z) = \frac{1}{w^2}\left(\frac{Z}{Z^2 + w^2} - \frac{1}{w}\ \arctan\frac{w}{Z}\right)$$
$$= \frac{-Z}{Z^2 + w^2}\left(\frac{2}{3}\ \frac{1}{Z^2 + w^2} + \frac{2\cdot 4}{3\cdot 5}\ \frac{w^2}{(Z^2 + w^2)^2} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\ \frac{w^4}{(Z^2 + w^2)^3} + \ldots\right),$$

$$B7(p, s) = \frac{1}{\sqrt{p}}\ \tan\sqrt{p}\,s,$$

$$B8(w^2, Z) = \frac{1}{w^2}\left(6\ B6\ (w^2,\ Z) + \frac{4Z}{(Z^2 + w^2)^2}\right)$$
$$= \frac{-16\ Z}{Z^2 + w^2}\left(\frac{1}{5}\ \frac{1}{Z^2 + w^2} + \frac{6}{5\cdot 7}\ \frac{w^2}{(Z^2 + w^2)^2} + \frac{6\cdot 8}{5\cdot 7\cdot 9}\ \frac{w^4}{(Z^2 + w^2)^3} + \ldots\right).$$

Remarks. For the two methods (central and focal), all the special functions B ought to be programmed with care, in such a way as to guarantee values possessing the maximum precision of the ordinator utilized; in order to do this, it is necessary, according to the value of the parameters, to utilize, one after the other, the standard trigonometric functions or the Taylor series corresponding to special functions.

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